

# TORSION IN FREE CENTRE-BY-NILPOTENT-BY-ABELIAN LIE RINGS OF RANK 2

RALPH STÖHR

ABSTRACT. For  $c \geq 2$ , the free centre-by-(nilpotent-of-class- $c-1$ )-by abelian Lie ring on a set  $X$  is the quotient  $L/[(L')^c, L]$  where  $L$  is the free Lie ring on  $X$ , and  $(L')^c$  denotes the  $c$ th term of the lower central series of the derived ideal  $L' = L^2$  of  $L$ . In this paper we give a complete description of the torsion subgroup of its additive group in the case where  $|X| = 2$  and  $c$  is a prime number.

## 1. INTRODUCTION

For an integer  $c \geq 2$ , the free centre-by-(nilpotent-of-class  $c - 1$ )-by-abelian Lie ring on a set  $X$  is the quotient

$$(1.1) \quad L/[(L')^c, L]$$

where  $L$  is an (absolutely) free Lie ring on  $X$  and  $(L')^c$  is the  $c$ th term of the lower central series of the derived ideal  $L' = L^2$  of  $L$ . In this note we determine the torsion subgroup of the additive group of (1.1) in the case where  $L$  has rank 2, that is  $X$  is a set of two elements, and  $c$  is a prime number. It is easily observed that for any  $c \geq 2$  and any  $X$  with  $|X| \geq 2$  the torsion subgroup of (1.1) is contained in the central ideal  $(L')^c/[(L')^c, L]$ . If  $c = p$ , where  $p$  is a prime, then the ideal  $(L')^p/[(L')^p, L]$  is a direct sum of a free abelian group and an infinite elementary abelian  $p$ -group. We exhibit an explicit  $\mathbb{Z}_p$ -basis of the torsion part if  $|X| = 2$ . This is the main result of this paper, see Theorem 5.1 in Section 5. In the earlier paper [1] a complete description of the torsion subgroup of the additive group of the more reduced quotient

$$(1.2) \quad L/([(L')^c, L] + L''')$$

was obtained, again in the case of rank 2, but for arbitrary  $c \geq 2$ . Note that (1.2) coincides with (1.1) for  $c = 2$  and  $c = 3$ , but not for  $c \geq 4$ . We make essential use of the results from [1].

The interest in torsion in relatively free Lie rings of the form (1.1) as well as their group-theoretic counterparts, the relatively free groups  $F/[\gamma_c(F'), F]$ , where  $F$  is a free group and  $\gamma_c(F')$  is the  $c$ th term of the lower central series of the derived subgroup  $F'$ , has a long history. If  $c = 2$ , these turn into the free centre-by-metabelian Lie rings and the free centre-by-metabelian groups, respectively, and it was the latter in which Kanta Gupta [8] first discovered torsion elements, a major

---

*Date:* January 11, 2017.

*2010 Mathematics Subject Classification.* Primary 17B01, 17B55.

surprise at the time. Later torsion was detected in  $F/[\gamma_c(F'), F]$  if  $c$  is a prime [15] and if  $c = 4$  [16]. Quite surprising, though, it turned out that  $F/[\gamma_c(F'), F]$  is torsion free if  $c$  is divisible two distinct primes [10]. As to Lie rings, early work in [12] on the free centre-by-metabelian Lie rings  $L/[L'', L]$ , and in particular on torsion in the additive group, turned out in need of some rectification, and this was eventually accomplished in [14] and [11]. For larger values of  $c$ , Drensky [7] proved that for any prime  $p$  the free Lie ring  $L/[(L')^p, L]$  contains non-trivial multilinear elements of degree  $2p + 1$  which have order  $p$ .

The paper is organized as follows. In Section 2 we introduce notation and assemble a number of preliminary results on the homogeneous components of free Lie rings. These are further examined in Section 3 where we derive our main result on Lie powers of free modules in prime degree, postponing, however, one key ingredient on Lie powers of prime degree  $p$  over fields of characteristic  $p$  to Section 4. In the final Section 5 we exploit the results of the previous sections to derive our main result.

## 2. NOTATION AND SOME PRELIMINARIES

We write maps on the right and use left-normed notation for Lie products. Let  $A$  be a free abelian group. By  $L(A)$  we denote the free Lie ring on  $A$ . For a positive integer  $c$ , we let  $L^c(A)$  denote the degree  $c$  homogeneous component of  $L(A)$ , that is the span of all left normed simple Lie products  $[a_1, a_2, \dots, a_c]$  with  $a_i \in A$ . In particular,  $L^1(A) = A$ . The universal envelope of  $L(A)$  can be identified with the tensor ring  $T(A) = \bigoplus_{c \geq 0} T^c(A)$  where  $T^0(A) = \mathbb{Z}$  and, for  $c > 0$ ,  $T^c(A) = \underbrace{A \otimes \cdots \otimes A}_c$ , the  $c$ th tensor power of  $A$ . By  $A^c$  we denote the  $c$ th symmetric power of  $A$ . The free metabelian Lie ring on  $A$  is the quotient of  $L(A)$  by its second derived ideal:  $M(A) = L(A)/L(A)''$ . This too is a graded Lie ring and we let  $M^c(A)$  denote its  $c$ th homogeneous component, that is  $M^c(A) = L^c(A)/(L^c(A) \cap L(A)'')$ . We call  $L^c(A)$  and  $M^c(A)$  the  $c$ th free Lie power and the  $c$ th free metabelian Lie power of  $A$ , respectively. It is well-known that if  $\mathcal{A}$  is an ordered  $\mathbb{Z}$ -basis of  $A$ , then the left normed simple Lie products  $[b_1, b_2, \dots, b_c]$  with  $b_i \in \mathcal{A}$  and  $b_1 > b_2 \leq \cdots \leq b_c$  form a  $\mathbb{Z}$ -basis of  $M^c(A)$  (see [2, Section 4.2.2]).

The canonical embedding of  $L(A)$  into its universal envelope  $T(A)$  induces in each degree  $c$  an embedding  $\nu_c : L^c(A) \rightarrow T^c(A)$ . By a well-known theorem of Wever (see [13, Chapter 5, Theorem 5.16]), the composite of this embedding with the natural projection  $\rho_c : T^c(A) \rightarrow L^c(A)$  defined by  $a_1 \otimes \cdots \otimes a_c \mapsto [a_1, \dots, a_c]$  amounts to multiplication by  $c$  on  $L^c(A)$ :

$$(2.1) \quad L^c(A) \xrightarrow{\nu_c} T^c(A) \xrightarrow{\rho_c} L^c(A), \quad \nu_c \rho_c = c.$$

The definition of  $M^c(A)$  gives rise to a short exact sequence

$$(2.2) \quad 0 \rightarrow B^c(A) \rightarrow L^c(A) \xrightarrow{\eta_c} M^c(A) \mapsto 0$$

where  $B^c(A) = L^c(A) \cap L(A)''$  and  $\eta_c$  is the natural projection map. Moreover, for  $c \geq 2$  the metabelian Lie power  $M^c(A)$  fits into a short exact sequence

$$(2.3) \quad 0 \rightarrow M^c(A) \xrightarrow{\mu_c} A \otimes A^{c-1} \xrightarrow{\kappa_c} A^c \rightarrow 0$$

where the maps  $\mu_c$  and  $\kappa_c$  are given by

$$[a_1, a_2, \dots, a_c] \mapsto a_1 \otimes (a_2 \circ \dots \circ a_c) - a_2 \otimes (a_1 \circ \dots \circ a_c)$$

and  $a_1 \otimes (a_2 \circ \dots \circ a_c) \mapsto a_1 \circ a_2 \circ \dots \circ a_c$ , respectively (see [9, Corollary 3.2]). Moreover, there is a map  $\lambda_c : A \otimes A^{c-1} \rightarrow M^c(A)$ , given by

$$a_1 \otimes (a_2 \circ \dots \circ a_c) \mapsto [a_1, a_2, a_3, \dots, a_c] + [a_1, a_3, a_2, \dots, a_c] + \dots + [a_1, a_c, a_2, \dots, a_{c-1}],$$

such that the composite of  $\mu_c$  and  $\lambda_c$  amounts to multiplication by  $c$  on  $M^c(A)$ :

$$(2.4) \quad M^c(A) \xrightarrow{\mu_c} A \otimes A^{c-1} \xrightarrow{\lambda_c} M^c(A), \quad \mu_c \lambda_c = c$$

see [1, Section 3]. Finally, for  $c \geq 2$  there is a map  $\theta_c : M^c(A) \rightarrow L^c(A)$  given by

$$(2.5) \quad [a_1, \dots, a_c] \mapsto \frac{1}{c} \left( \sum_{\sigma} [a_1, a_{\pi(2)}, \dots, a_{\pi(c)}] - \sum_{\tau} [a_2, a_{\tau(1)}, \dots, a_{\tau(c)}] \right),$$

where the sums run over all permutations  $\sigma$  of  $\{2, 3, \dots, n\}$  and all permutations  $\tau$  of  $\{1, 3, \dots, n\}$ , respectively. Although we work over  $\mathbb{Z}$ , the factor  $1/c$  in (2.5) makes sense as the expression on the right hand side can be written as a Lie polynomial with integer coefficients (see [4, Section 2]). This Lie polynomial has been calculated in [5, Proposition 7.3], but since it is rather involved, we prefer to use the compact form (2.5) in what follows. The composite of  $\theta_c$  and the natural projection  $\eta_c$  as in (2.2) amounts to multiplication by  $(c-2)!$  on  $M^c(A)$ :

$$(2.6) \quad M^c(A) \xrightarrow{\theta_c} L^c(A) \xrightarrow{\eta_c} M^c(A), \quad \theta_c \eta_c = (c-2)!$$

(see [4, Section 2]).

Now suppose that  $A$  carries the structure of a module for the polynomial ring  $U = \mathbb{Z}[X]$  where  $X$  is a finite set of variables. Then all the objects introduced in this section such as Lie powers, symmetric powers etc. will be regarded as  $U$ -modules under the derivation action. For example, for  $x \in X$ ,  $a_i \in A$ ,

$$[a_1, a_2, \dots, a_c]x = \sum_{i=1}^c [a_1, \dots, a_i x, \dots, a_c],$$

Note that all the maps introduced in this section are compatible with the derivation action, that is, all these maps are, in fact,  $\mathbb{Z}[X]$ -module homomorphisms. This will be used in what follows without further reference being given. The ring of integers  $\mathbb{Z}$  will be regarded as a trivial  $U$ -module.

### 3. LIE POWERS OF FREE MODULES

In this section we retain the notation introduced in Section 2, but now we assume throughout that  $A$  is a *free*  $U$ -module for the polynomial ring  $U = \mathbb{Z}(X)$ . All homology groups in this section will be over the ground ring  $U$ . For brevity, if  $W$  is a  $U$ -module, the homology groups  $H_k(U, W) = \text{Tor}_k^U(W, \mathbb{Z})$ ,  $k \geq 0$ , will be written as  $H_k(W)$ .

It is well known (see, for example, [14, Lemma 5.2]) that if  $A$  is a free  $U$ -module then both  $T^c(A)$  and  $A \otimes A^{c-1}$  with  $c \geq 2$  are also free  $U$ -modules under the derivation action.

**Lemma 3.1.** *Let  $A$  be a free  $U$ -module,  $c \geq 2$ . Then*

- (i) *the tensor products  $L_c(A) \otimes_U \mathbb{Z}$  and  $M_c(A) \otimes_U \mathbb{Z}$  are direct sums of a free abelian group and a torsion group of exponent dividing  $c$ ,*
- (ii) *for  $k \geq 1$  the homology groups  $H_k(L_c(A))$  and  $H_k(M_c(A))$  are torsion groups of exponent dividing  $c$ .*

*Proof.* By applying the homology functor to the maps in (2.1) we get that  $H_k(\nu_c \rho_c)$  is multiplication by  $c$  on  $H_k(L_c(A))$  for all  $k \geq 0$ . Since  $A$  is a free  $U$ -module,  $H_0(T_c(A))$  is free abelian and  $H_k(T_c(A)) = 0$  for  $k \geq 1$ . Then, for any  $u \in H_k(L^c A)$  with  $k > 0$ ,

$$cu = uH_k(\nu_c \rho_c) = (uH_k(\nu_c))H_k(\rho_c) = 0H_k(\rho_c) = 0.$$

The same holds if  $k = 0$  and  $u \in \text{Ker}(H_0(\nu_c))$ . Hence multiplication by  $c$  annihilates both the homology groups  $H_k(L_c(A))$  for  $k \geq 1$  and the kernel of the homomorphism  $H_0(\nu_c)$ . The image of this homomorphism is contained in the free abelian group  $H_0(T_c(A))$ , and hence itself free abelian. The results (i) and (ii) for  $L_c(A)$  follow. The proof of (i) and (ii) for  $M_c(A)$  are obtained by a similar argument using the maps in (2.4) instead of those in (2.1).  $\square$

Now we consider Lie powers of prime degree. Let  $p$  be a prime. By applying the homology functor to the short exact sequence (2.2) we obtain the exact sequence

$$(3.1) \quad \cdots \rightarrow H_1(M^p(A)) \rightarrow B^p(A) \otimes_U \mathbb{Z} \rightarrow L^p(A) \otimes_U \mathbb{Z} \xrightarrow{\eta_p \otimes 1} M^p(A) \otimes_U \mathbb{Z} \rightarrow 0.$$

By Lemma 3.1.,  $L^p(A) \otimes_U \mathbb{Z}$  is a direct sum of a free abelian group and an elementary abelian  $p$ -group, and  $H_1(M^p(A))$  is an elementary abelian  $p$ -group. Note that this does not exclude the possibility that these torsion subgroups are trivial. Now the exact sequence (3.1) yields that  $B^p(A) \otimes_U \mathbb{Z}$  is a direct sum of a free abelian group and a (possibly trivial)  $p$ -group of exponent dividing  $p^2$ . In fact, we will show that this group has actually no torsion, in other words, we will prove the following result.

**Lemma 3.2.** *Let  $A$  be a free  $U$ -module and  $p$  a prime. Then the tensor product  $B^p(A) \otimes_U \mathbb{Z}$  is a free abelian group.*

*Proof.* Since we already know that  $B^p(A) \otimes_U \mathbb{Z}$  is a direct sum of a free abelian group and a  $p$ -group of finite exponent, it is sufficient to show that no non-zero element in  $B^p(A) \otimes_U \mathbb{Z}$  is annihilated by  $p$ . We use reduction modulo  $p$ , that is the short exact sequence

$$0 \rightarrow B^p(A) \xrightarrow{p} B^p(A) \rightarrow B^p(A) \otimes \mathbb{Z}_p \rightarrow 0$$

which, in its turn, gives rise to the exact sequence

$$(3.2) \quad \cdots \rightarrow H_1(B^p(A) \otimes \mathbb{Z}_p) \rightarrow B^p(A) \otimes_U \mathbb{Z} \xrightarrow{p} B^p(A) \otimes_U \mathbb{Z} \rightarrow B^p(A) \otimes_U \mathbb{Z}_p \rightarrow 0.$$

The Lemma will be proved once we show that the homology group on the left is zero, and this will certainly follow if we can verify that  $B^p(A) \otimes \mathbb{Z}_p$ , regarded as a

module for the polynomial ring  $\mathbb{Z}_p[X]$ , is projective. The proof of this fact will be given in the next section (see Corollary 4.2). This will then complete the proof of Lemma 3.2.  $\square$

Now we have all the ingredients in place to prove the main result of this section. Recall the homomorphism  $\theta_c : M^c(A) \rightarrow L^c(A)$  defined by (2.5).

**Proposition 3.3.** *Let  $A$  be a free  $U$ -module and  $p$  a prime. Then the torsion subgroups of  $L^p(A) \otimes_U \mathbb{Z}$  and  $M^p(A) \otimes_U \mathbb{Z}$  are isomorphic, and the homomorphism  $\theta_p \otimes 1$  maps the latter isomorphically onto the former.*

*Proof.* By Lemma 3.1(i) both  $L^p(A) \otimes_U \mathbb{Z}$  and  $M^p(A) \otimes_U \mathbb{Z}$  are direct sums of a free abelian group and an elementary abelian  $p$ -group. Consider the maps in (2.6). By trivializing the  $U$ -action we obtain homomorphisms

$$M^p(A) \otimes_U \mathbb{Z} \xrightarrow{\theta_p \otimes 1} L^p(A) \otimes_U \mathbb{Z} \xrightarrow{\eta_p \otimes 1} M^p(A) \otimes_U \mathbb{Z}, \quad \theta_k \eta_k \otimes 1 = (p-2)!$$

So the restriction of the composite  $\theta_k \eta_k \otimes 1$  to the torsion subgroup of the tensor product  $M^p(A) \otimes_U \mathbb{Z}$ , an elementary abelian  $p$ -group, is multiplication by  $(p-2)!$ , that is, it is an isomorphism. It follows that the homomorphism  $\theta_p \otimes 1$  maps the torsion subgroup of  $M^p(A) \otimes_U \mathbb{Z}$  isomorphically into the torsion subgroup of  $L^p(A) \otimes_U \mathbb{Z}$ . To prove the proposition, we need to verify that this map is also surjective, that is, the homomorphism  $\theta_k \otimes 1$  maps the torsion subgroup of  $M^p(A) \otimes_U \mathbb{Z}$  isomorphically onto the torsion subgroup of  $L^p(A) \otimes_U \mathbb{Z}$ . But this is true since otherwise the restriction of  $\eta_p \otimes 1$  to the torsion subgroup of  $L^p(A) \otimes_U \mathbb{Z}$  would have a non-trivial kernel. This, however, is not the case, as follows from the exactness of (3.1). Since  $B^p(A) \otimes_U \mathbb{Z}$  is free abelian by Lemma 3.2, and  $H_1(M^p(A))$  is torsion by Lemma 3.1(ii), there cannot be any torsion elements in the kernel of  $\eta_p \otimes 1$ . This proves the proposition.  $\square$

In the next section we fill in the gap left in the proof of Lemma 3.2.

#### 4. THE DEGREE $p$ LIE POWER IN CHARACTERISTIC $p$

In this section  $V$  denotes a vector space over a field  $K$  of prime characteristic  $p$ . Moreover, we will assume that  $V$  is a module for the polynomial ring  $K[X]$  where  $X$  is a finite set of indeterminates. Otherwise we will use all the notation introduced in Section 2, in particular,  $L^p(V)$ ,  $M^p(V)$  and  $T^p(V)$  are the  $p$ th Lie, metabelian Lie, and tensor powers of  $V$ , respectively,  $B^p(V) = L^p(V) \cap L(V)''$  is the kernel of the natural projection  $L^p(V) \rightarrow M^p(V)$ , and all of these will be regarded as  $K[X]$ -modules under the derivation action. Recall that  $L^p(V)$  may be regarded as a submodule of the tensor power  $T^p(V)$ , and hence  $B^p(V)$  is also a submodule of  $T^p(V)$ .

**Lemma 4.1.** *The submodule  $B^p(V)$  is a direct summand of the  $K[X]$ -module  $T^p(V)$ .*

This result is essentially proved as Theorem 3.1 in [6], except that there the module  $V$  is assumed to be finite-dimensional. In what follows we reproduce the

proof from [6] with some minor amendments necessary to accommodate infinite dimensional modules.

*Proof.* For each  $r \geq 1$  choose a basis  $\mathcal{B}^{(r)}$  of  $L^r(V)$  and let  $\mathcal{B} = \bigcup_r \mathcal{B}^{(r)}$ . Thus  $\mathcal{B}$  is a basis of  $L(V)$ . For  $b \in \mathcal{B}$ , let  $\deg(b)$  denote the degree of  $b$ , that is,  $\deg(b) = r$  for  $b \in \mathcal{B}^{(r)}$ . Order  $\mathcal{B}$  in any way subject to  $b < b'$  whenever  $\deg(b) < \deg(b')$ . By the Poincaré–Birkhoff–Witt Theorem,  $T^p(V)$  has a basis  $\mathcal{C}$  consisting of all products of the form  $b_1 \otimes b_2 \otimes \cdots \otimes b_k$  with  $b_1, \dots, b_k \in \mathcal{B}$ ,  $b_1 \leq b_2 \leq \cdots \leq b_k$  and  $\deg(b_1) + \cdots + \deg(b_k) = p$ . More specifically, any basis element  $c \in \mathcal{C}$  has the form

$$(4.1) \quad c = b_1^{(1)} \otimes \cdots \otimes b_{k_1}^{(1)} \otimes b_1^{(2)} \otimes \cdots \otimes b_{k_2}^{(2)} \otimes \cdots \otimes b_1^{(p)} \otimes \cdots \otimes b_{k_p}^{(p)},$$

where  $k_1, \dots, k_p$  are non-negative integers such that  $k_1 + 2k_2 + \cdots + pk_p = p$  and where, for  $i = 1, \dots, p$ , we have  $b_1^{(i)}, \dots, b_{k_i}^{(i)} \in \mathcal{B}^{(i)}$  and  $b_1^{(i)} \leq \cdots \leq b_{k_i}^{(i)}$ . We call the  $p$ -tuple  $(k_1, \dots, k_p)$  the *type* of  $c$  and denote it by  $\text{type}(c)$ . Let  $\Omega$  denote the set of all such types. We order  $\Omega$  (lexicographically) by  $(k_1, \dots, k_p) > (k'_1, \dots, k'_p)$  if for some  $j \in \{1, \dots, p\}$  we have  $k_i = k'_i$  for all  $i < j$  but  $k_j > k'_j$ . Using this ordering, write  $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$  where  $\omega_1 > \omega_2 > \cdots > \omega_m$ . Thus  $\omega_1 = (p, 0, \dots, 0)$  and  $\omega_m = (0, \dots, 0, 1)$ .

For  $i = 1, \dots, m$ , define  $\mathcal{C}_i = \{c \in \mathcal{C} : \text{type}(c) = \omega_i\}$  and let  $X_i$  denote the subspace of  $T^p(V)$  spanned by  $\mathcal{C}_i \cup \mathcal{C}_{i+1} \cup \cdots \cup \mathcal{C}_m$ . Also, write  $X_{m+1} = 0$ . Thus

$$T^p(V) = X_1 > X_2 > \cdots > X_m > X_{m+1} = 0.$$

Note that  $X_i/X_{i+1}$  has basis  $\mathcal{C}_i$  modulo  $X_{i+1}$ . Furthermore,  $\mathcal{C}_1$  consists of all products  $b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_p^{(1)}$  with  $b_1^{(1)}, \dots, b_p^{(1)} \in \mathcal{B}^{(1)}$  and  $b_1^{(1)} \leq \cdots \leq b_p^{(1)}$ . Also,  $\mathcal{C}_m = \mathcal{B}^{(p)}$  and  $X_m = L^p(V)$ .

Let  $i \in \{1, \dots, m\}$  where  $\omega_i = (k_1, \dots, k_p)$ . For  $c \in \mathcal{C}_i$  written as in (4.1) it is well known and easy to verify that the value of  $c$  modulo  $X_{i+1}$  is unchanged by any permutation of the factors  $b_1^{(1)}, \dots, b_{k_p}^{(p)}$ . In particular, for all  $\pi_1 \in \text{Sym}(k_1), \dots, \pi_p \in \text{Sym}(k_p)$ , we have

$$(4.2) \quad b_{\pi_1(1)}^{(1)} \otimes \cdots \otimes b_{\pi_1(k_1)}^{(1)} \otimes \cdots \otimes b_{\pi_p(1)}^{(p)} \otimes \cdots \otimes b_{\pi_p(k_p)}^{(p)} + X_{i+1} = c + X_{i+1}.$$

It follows easily that  $X_i$  is a  $K[X]$ -submodule of  $T^p(V)$ . For  $c$  written as before let  $\bar{c} \in S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V))$  be defined by

$$\bar{c} = (b_1^{(1)} \circ \cdots \circ b_{k_1}^{(1)}) \otimes (b_1^{(2)} \circ \cdots \circ b_{k_2}^{(2)}) \otimes \cdots \otimes (b_1^{(p)} \circ \cdots \circ b_{k_p}^{(p)}).$$

Clearly  $\{\bar{c} : c \in \mathcal{C}_i\}$  is a basis of  $S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V))$ . Furthermore, it follows easily from (4.2) that the linear map given by  $c + X_{i+1} \mapsto \bar{c}$  is a  $K[X]$ -module isomorphism from  $X_i/X_{i+1}$  to  $S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V))$ . Thus

$$(4.3) \quad X_i/X_{i+1} \cong S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V)).$$

Suppose that  $i \in \{2, \dots, m-1\}$  where  $\omega_i = (k_1, \dots, k_p)$ . Thus  $k_1, \dots, k_p < p$  (and, in fact,  $k_p = 0$ ). Let  $\sigma_i : S^{k_1}(L^1(V)) \otimes \cdots \otimes S^{k_p}(L^p(V)) \rightarrow T^p(V)$  be the linear map defined on the basis  $\{\bar{c} : c \in \mathcal{C}_i\}$  by

$$\sigma_i(\bar{c}) = \frac{1}{k_1! \cdots k_p!} \sum_{\substack{\pi_1 \in \text{Sym}(k_1), \dots, \\ \pi_p \in \text{Sym}(k_p)}} b_{\pi_1(1)}^{(1)} \otimes \cdots \otimes b_{\pi_1(k_1)}^{(1)} \otimes \cdots \otimes b_{\pi_p(1)}^{(p)} \otimes \cdots \otimes b_{\pi_p(k_p)}^{(p)},$$

where  $c$  is written as in (4.1). It is straightforward to verify that  $\sigma_i$  is a  $K[X]$ -module homomorphism. It follows easily from (4.2) that  $\sigma_i(\bar{c}) \in X_i$  and, indeed,  $\sigma_i(\bar{c}) + X_{i+1} = c + X_{i+1}$  for all  $c \in \mathcal{C}_i$ . Thus the map  $\sigma_i$  is injective and we have  $X_i = \text{Im}(\sigma_i) \oplus X_{i+1}$ . Since  $X_m = L^p(V)$  we therefore have

$$X_2 = \text{Im}(\sigma_2) \oplus \cdots \oplus \text{Im}(\sigma_{m-1}) \oplus L^p(V).$$

Let  $W$  be the submodule of  $X_2$  given by

$$(4.4) \quad W = \text{Im}(\sigma_2) \oplus \cdots \oplus \text{Im}(\sigma_{m-1}) \oplus (L(V)'' \cap L^p(V)).$$

Consider the maps

$$\alpha : T^p(V) \rightarrow V \otimes V^{p-1}, \quad \beta : V \otimes V^{p-1} \rightarrow T^p(V)$$

defined by

$$a_1 \otimes a_2 \otimes \cdots \otimes a_p \mapsto a_1 \otimes (a_2 \circ \cdots \circ a_p)$$

and

$$a_1 \otimes (a_2 \circ \cdots \circ a_p) \mapsto \frac{1}{(p-1)!} \sum_{\pi} a_1 \otimes a_{\pi(2)} \otimes \cdots \otimes a_{\pi(p)},$$

where  $a_i \in V$  and  $\pi$  runs over all permutations of  $\{2, 3, \dots, p\}$ . Then  $\alpha$  is surjective and the composite  $\beta\alpha$  is the identity map on  $V \otimes V^{p-1}$ . Hence we have a direct decomposition

$$T^p(V) = \text{Ker}(\alpha) \oplus \text{Im}(\beta) \cong \text{Ker}(\alpha) \oplus (V \otimes V^{p-1}).$$

We claim that  $\text{Ker}(\alpha) = W$ . It is easily seen that  $W$  is contained in  $\text{Ker}(\alpha)$ . To verify that we have actually equality, note that the elements

$$(4.5) \quad b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_p^{(1)} \quad \text{with } b_1^{(1)}, \dots, b_p^{(1)} \in \mathcal{B}^{(1)} \text{ and } b_1^{(1)} \leq \cdots \leq b_p^{(1)}$$

form a basis of  $T^p(V)$  modulo  $X_2$ . Furthermore, the Lie products

$$(4.6) \quad [b_1^{(1)}, b_2^{(1)}, \dots, b_p^{(1)}] \quad \text{with } b_1^{(1)}, \dots, b_p^{(1)} \in \mathcal{B}^{(1)} \text{ and } b_1^{(1)} > b_2^{(1)} \leq \cdots \leq b_p^{(1)}$$

form a basis of  $L^p(V)$  modulo  $L^p(V) \cap L(V)''$  (see Section 2). It follows that the elements (4.5) together with the elements (4.6) form a basis of  $T^p(V)$  modulo  $W$ . Moreover, the images of these elements under the map  $\alpha$  form a basis of  $V \otimes V^{p-1}$ . Indeed, we have

$$(4.7) \quad (b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_p^{(1)})\alpha = b_1^{(1)} \otimes (b_2^{(1)} \circ \cdots \circ b_p^{(1)})$$

and

$$(4.8) \quad ([b_1^{(1)}, b_2^{(1)}, \dots, b_p^{(1)}])\alpha = b_1^{(1)} \otimes (b_2^{(1)} \circ \cdots \circ b_p^{(1)}) - b_2^{(1)} \otimes (b_1^{(1)} \circ \cdots \circ b_p^{(1)}).$$

This can easily be seen from the short exact sequence (2.3) (with  $V$  instead of  $A$ ). Indeed, the elements (4.7) are mapped by  $\kappa_p$  one-to-one onto the canonical basis of  $V^p$ , and the elements (4.8) are precisely the images of the canonical basis elements of  $M^p(V)$  under the map  $\mu_p$ . Consequently, we have the desired equality  $\text{Ker}(\alpha) = W$ , and so  $T^p(V) = W \oplus \text{Im}(\beta)$ . By (4.4),  $L^p(V) \cap L(V)''$  is a direct summand of  $W$ . Thus  $L^p(V) \cap L(V)''$  is a direct summand of  $T^p(V)$  and we have Lemma 4.1.  $\square$

Now the result we need to complete the proof of Proposition 3.3 follows easily.

**Corollary 4.2.** *If  $V$  is a free  $K[X]$ -module, then  $B^p(V)$  is a projective  $K[X]$ -module.*

*Proof.* If  $V$  is a free  $K[X]$ -module, then the tensor power  $T^p(V)$  is also a free  $K[X]$ -module, see [14, Lemma 5.2]). Since  $B^p(V)$  is a direct summand of  $T^p(V)$ , it is projective.  $\square$

## 5. THE MAIN RESULT

In this Section  $L$  denotes a free Lie ring of finite rank with free generating set  $X$ . Our aim is to determine the torsion subgroup of the additive group of the quotient (1.1). In view of the short exact sequence

$$0 \rightarrow (L')^c / [(L')^c, L] \rightarrow L / [(L')^c, L] \rightarrow L / (L')^c \rightarrow 0,$$

this is a free central extension of the free (nilpotent-of-class- $c-1$ )-by-abelian Lie ring  $L / (L')^c$ . The additive structure of the latter is well-understood. Its underlying abelian group is free abelian [3]. Consequently, any torsion elements must be contained in the central quotient  $(L')^c / [(L')^c, L]$ , and it is this quotient we will focus on from now on. By the Shirshov-Witt Theorem, the derived ideal  $L'$  is itself a free Lie ring, namely, the free Lie ring on  $L' / L''$ :  $L' = L(L' / L'')$ . This is a graded Lie ring and its degree  $c$  homogeneous component  $L^c(L' / L'')$  is isomorphic to the lower central quotient

$$(5.1) \quad L^c(L' / L'') \cong (L')^c / (L')^{c+1}.$$

The adjoint representation induces on these lower central quotients the structure of an  $L / L'$ -module, and hence of a module for its universal envelope  $U = U(L / L')$ . The latter may be identified with the polynomial ring on  $X$ :  $U = \mathbb{Z}[X]$ . Thus (5.1) is actually a  $U$ -module isomorphism. In view of the canonical isomorphism

$$((L')^c / (L')^{c+1} \otimes_U \mathbb{Z} \cong (L')^c / [(L')^c, L],$$

trivializing the  $U$ -action on both sides of (4.1) gives an isomorphism

$$(5.2) \quad (L')^c / [(L')^c, L] \cong L^c(L' / L'') \otimes_U \mathbb{Z}.$$

We will exploit this isomorphism to investigate the additive structure of the quotient on the left hand side by examining the tensor product on the right hand side.

Suppose that  $L$  has rank 2 and, say,  $X = \{x, y\}$ . Then  $L' / L''$  is a free cyclic module over the polynomial ring  $U = \mathbb{Z}[x, y]$  with free generator  $[y, x]$ , see [1, Proof of Theorem 6.1]. If  $c$  is a prime, say  $c = p$ , then Proposition 3.3 applies to the tensor product on the right hand side of (5.2). Hence this tensor product is a direct sum of a free abelian group and an elementary abelian  $p$ -group. Moreover, the torsion subgroup is the image in  $L^p(L' / L'') \otimes_U \mathbb{Z}$  of the torsion subgroup of  $M^p(L' / L'') \otimes_U \mathbb{Z}$  under the map  $\theta_p \otimes 1$ . A complete description of the latter is given in [1, Corollary 6.2]. The elements

$$[[u, y], [u, x], \underbrace{u, \dots, u}_{p-2}] \otimes 1$$



where  $u = [y, x, \underbrace{x, \dots, x}_s, \underbrace{y, \dots, y}_t]$  with  $s, t \geq 0$  form a basis of this torsion subgroup as a  $\mathbb{Z}_p$ -module. Applying  $\theta_p \otimes 1$  to such a basis element gives

$$\frac{(p-2)!}{p} \sum_{i=0}^{p-1} ([u, y], \underbrace{u, \dots, u}_i, [u, x], \underbrace{u, \dots, u}_{p-2-i}) - ([u, x], \underbrace{u, \dots, u}_i, [u, y], \underbrace{u, \dots, u}_{p-2-i}) \otimes 1.$$

Since this is an element of order  $p$  we can drop the factor of  $(p-2)!$  in the statement of our main result, which summarizes the discussion in this final section.

**Theorem 5.1.** *Let  $L$  be a free Lie ring of rank 2 with free generators  $x$  and  $y$ , and let  $p$  be a prime. Then the quotient  $(L')^p / [(L')^p, L]$  is a direct sum of a free abelian group and an elementary abelian  $p$ -group. Modulo  $[(L')^p, L]$  the elements*

$$\frac{1}{p} \sum_{i=0}^{p-1} ([u, y], \underbrace{u, \dots, u}_i, [u, x], \underbrace{u, \dots, u}_{p-2-i}) - ([u, x], \underbrace{u, \dots, u}_i, [u, y], \underbrace{u, \dots, u}_{p-2-i})$$

where  $u = [y, x, \underbrace{x, \dots, x}_s, \underbrace{y, \dots, y}_t]$  with  $s, t \geq 0$  form a basis of this torsion subgroup as a  $\mathbb{Z}_p$ -module. □

The legality of the factor  $1/p$  in the statement of the theorem is explained in Section 2.

## REFERENCES

- [1] Maria Alexandrou and Ralph Stöhr, Free centre-by-nilpotent-by-abelian Lie rings of rank 2, *J. Austral. Math. Soc.*, Published online: 07 May 2015, pp. 1-11. DOI: <https://doi.org/10.1017/S1446788715000051>
- [2] Yu. A. Bakhturin, *Identical relations in Lie algebras*, Nauka, Moscow, 1985 (Russian). English translation: VNU Science Press, Utrecht, 1987.
- [3] L.A. Bokut', A basis of free polynilpotent Lie algebras (Russian), *Algebra i Logika* **2**, no. 4 (1963), 13-18,
- [4] R.M.Bryant, L.G. Kovács and Ralph Stöhr, Lie powers of modules for groups of prime order, *Proc. London Math. Soc.* **84** (2002), 334–374.
- [5] R.M.Bryant and Ralph Stöhr, On the module structure of free Lie algebras, *Trans. Amer. Math. Soc.* **352** (2000), 901–934.
- [6] R.M.Bryant and Ralph Stöhr, Lie powers in prime degree, *Q. J. Math.* **56** (2005), 473-489.
- [7] Vesselin Drensky, Torsion in the additive group of relatively free Lie rings, *Bull. Austral. Math. Soc.* **33** (1986), no. 1, 81–87.
- [8] Chander Kanta Gupta, The free centre-by-metabelian groups, *J. Austral. Math. Soc.* **16** (1973), 294–299.
- [9] T. Hannebauer and R. Stöhr, Homology of groups with coefficients in free metabelian Lie powers and exterior powers of relation modules and applications to group theory, in *Proc. Second Internat. Group Theory Conf., Bressanone, 1989, Rend. Circ. Mat. Palermo (2) Suppl.* **23** (1990), 77–113.
- [10] Marianne Johnson and Ralph Stöhr, Free central extensions of groups and modular Lie powers of relation modules, *Proc. Amer. Math. Soc.*, **138** (2010), no. 11, 3807–3814.
- [11] L.G. Kovács and Ralph Stöhr, Free centre-by-metabelian Lie algebras in characteristic 2, *Bull.Lond. Math. Soc.*, **46** (2014), 491–502.
- [12] Yu. V. Kuz'min, Free center-by-metabelian groups, Lie algebras and  $\mathcal{D}$ -groups (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), no. 1, 3–33, 231. English translation: *Math. USSR Izvestija* **11** (1977), no. 1, 1–30.

- [13] Magnus, W., Karrass, A, Solitar, D. *Combinatorial Group Theory*, Wiley-Interscience, New York, 1966.
- [14] Nil Mansuroğlu and Ralph Stöhr, Free centre-by-metabelian Lie rings, *Q. J. Math.* **65** (2014), no. 2, 555–579.
- [15] Ralph Stöhr, On torsion in free central extensions of some torsion-free groups, *J. Pure Appl. Algebra* **46** (1987), no. 2-3, 249–289.
- [16] Ralph Stöhr, Homology of free Lie powers and torsion in groups, *Israel J. Math.* **84** (1993), 65–87.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MANCHESTER, ALAN TURING BUILDING, MANCHESTER, M13 9PL, UNITED KINGDOM

*E-mail address:* `Ralph.Stohr@manchester.ac.uk`